Chern-Simons-Rozansky-Witten topological field theory

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Abstract

We construct and study a new topological field theory in three dimensions. It is a hybrid between Chern-Simons and Rozansky-Witten theory and can be regarded as a topologically-twisted version of the $N = 4$ $d = 3$ supersymmetric gauge theory recently discovered by Gaiotto and Witten. The model depends on a gauge group G and a hyper-Kähler manifold X with a tri-holomorphic action of G. In the case when X is an affine space, we show that the model is equivalent to Chern-Simons theory whose gauge group is a supergroup. This explains the role of Lie superalgebras in the construction of Gaiotto and Witten. For general X, our model appears to be new. We describe some of its properties, focusing on the case when G is simple and X is the cotangent bundle of the flag variety of G. In particular, we show that Wilson loops are labeled by objects of a certain category which is a quantum deformation of the equivariant derived category of coherent sheaves on X.

1 Introduction and summary

Recently, an interesting new class of three-dimensional gauge theories with $N = 4$ supersymmetry has been discovered [1]. The distinguishing feature of these field theories is that the gauge field kinetic term is the Chern-Simons term, while the usual Yang-Mills kinetic term is absent.

The theories constructed in [1] contain a Chern-Simons gauge field interacting with $N = 4$ hypermultiplets. In the limit of vanishing gauge coupling hypermultiplets are described by an $N = 4$ $d = 3$ sigma-model whose target X is required to be hyper-Kähler by supersymmetry. For nonvanishing gauge coupling the theory can be regarded as a gauged sigma-model. If the target X is flat, the theory constructed in [1] is superconformal and has $SU(2)_N \times SU(2)_R$ R-symmetry with respect to which supercharges transform as $(2, 2)$. For general X, the model has $SU(2)_N$ R-symmetry with respect to which the supercharges transform as a doublet. In either case, one can twist $SU(2)_N$ R-symmetry to get a topological field theory. It is an unusual theory, since it straddles the boundary between Schwarz-type and Witten-type topological field theories. [1]

In this paper we study the topologically-twisted version of the Gaiotto-Witten theory. We find that in the case of flat X it is equivalent to the pure Chern-Simons theory [2, 3] whose gauge group is a supergroup. More precisely, the topologically-twisted Gaiotto-Witten theory is obtained from the supergroup Chern-Simons theory by gauge-fixing the odd part of the supergroup. The gauge group G of the Gaiotto-Witten theory is the residual (even) part of the supergroup. This provides a simple explanation of the fact that Gaiotto-Witten theories with flat X are in one-to-one correspondence with Lie superalgebras with nondegenerate invariant metric [1]. This observation also helps to construct BRST-invariant Wilson loop operators: they are naturally associated with finite-dimensional representations of the supergroup.

For general X the topologically twisted Gaiotto-Witten theory can be regarded as a gauged version of the Rozansky-Witten theory (the 3d topological sigma-model with target X constructed in [4]). Thus it is a hybrid of Chern-Simons and Rozansky-Witten theory. It is associated to a quadruple (G, κ, X, I), where G is a compact Lie group, κ is an invariant metric on its Lie algebra, X is a hyper-Kähler manifold with a tri-holomorphic action of G, and I is a complex structure on X such that the complex moment map with respect to the complex symplectic form Ω_I is isotropic with respect to κ.

[1]In retrospect, it is obvious that this dichotomy is not a good one: upon gauge-fixing Chern-Simons theory [2, 3] becomes a Witten-type topological theory. Rather, Schwarz-type theories are a subclass of Witten-type theories distinguished by the property that they have a unitary sector. In Chern-Simons theory this is the ghost-number zero sector.

For a simple G, the metric on the Lie algebra is unique up to a multiple. The most obvious choice of X in this case is the cotangent bundle of the flag variety $G_{\mathbb{C}}/B$, where B is a Borel subgroup of $G_{\mathbb{C}}$. It is well known that the complex moment map for the obvious G-action on $T^*(G_{\mathbb{C}}/B)$ is nilpotent. In fact, the image of this map is precisely the set of nilpotent elements in the Lie algebra of $G_{\mathbb{C}}$. The moment map is generically one-to-one and gives the so-called Springer resolution of the variety of nilpotent elements.[2]

Thus to any compact simple Lie group we can attach two natural 3d TFTs with Chern-Simons terms: the ordinary Chern-Simons theory and the Chern-Simons-Rozansky-Witten theory with target $T^*(G_{\mathbb{C}}/B)$. It is well known that the former theory is related to representations of quantum groups. In this paper we begin the study of the latter theory. Namely, we compute the algebra of local operators (which turns out to be rather trivial, as in Chern-Simons theory) and determine the category of Wilson loop operators. The category of Wilson loops turns out to be a novel deformation of the $G_{\mathbb{C}}$-equivariant derived category of coherent sheaves on $T^*(G_{\mathbb{C}}/B)$. We show that such a deformation can be defined in a rather general situation of a differential graded Poisson algebra with a Hamiltonian action of a Lie group, provided the moment map satisfies a certain constraint. In one special case we compute the braiding of Wilson loops.

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2 Construction of the Chern-Simons-Rozansky-Witten model.

2.1 Fields and BRST transformations

In this section we will construct the CSRW theory "from scratch", by postulating certain BRST transformations and then constructing a suitable BRST-invariant Lagrangian. It is shown in the appendix that the same theory can also be obtained by twisting the Gaiotto-Witten theory.

Let M be a Riemannian 3-manifold with local coordinates x^μ, $\mu = 1, 2, 3$. We are going to construct a gauged version of the Rozansky-Witten model with target X, where X is a hyper-

[2]We note that the Gaiotto-Witten model with gauge group $G = SU(N)$ and target $T^*(G_{\mathbb{C}}/B)$ previously appeared in the string theory context [5]. It is an $N = 4$ $d = 3$ field theory which describes the degrees of freedom living on the boundary of a stack of N D3-branes ending on a a bound state of k NS fivebranes and one D5-brane. Here k is the Chern-Simons level, and it is assumed that the D3-brane theory is in the vacuum where the gauge group $U(N)$ is broken down to its maximal torus $U(1)^N$.

Kähler manifold of complex dimension $\dim_{\mathbb{C}} X = 2n$ which admits an action of the group G. Let V_a for $a = 1, \ldots, \dim G$ be the vector fields on X corresponding to this G-action. We can view them as components of a section of $TX \otimes \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G. In local complex coordinates $\phi^{\hat{I}} = (\phi^I, \phi^{\bar{I}})$ with $I, \bar{I} = 1, \ldots, 2n$ we write these vector fields as

$$V_a = V_a^{\hat{I}} \partial_{\hat{I}} = V_a^I \partial_I + V_a^{\bar{I}} \partial_{\bar{I}}.$$

Thus under an infinitesimal G-transformation with parameters ϵ^a the bosonic fields $\phi^I, \phi^{\bar{I}}$ transform as

$$\delta_\epsilon \phi^I = \epsilon^a V_a^I, \quad \delta_\epsilon \phi^{\bar{I}} = \epsilon^a V_a^{\bar{I}}.$$

The vector fields V_a satisfy

$$[V_a, V_b] = f_{ab}^c V_c,$$

where f_{ab}^c are the structure constants of $\mathfrak{g}$.

In order for G to be a global symmetry of the RW model it is necessary and sufficient that for all a the $(1,0)$ vector field V_a^I be holomorphic and preserve the symplectic structure Ω_{IJ}. We will further assume that the G action preserves the Kähler form on X. This implies that locally there exist moment maps $\mu_+, \mu_-, \mu_3 : X \to \mathfrak{g}^*$, i.e. $\mathfrak{g}^*$-valued functions on X satisfying:

$$d\mu_{+\ a} = -i_{V_a}(\Omega), \qquad d\mu_{-\ a} = -i_{V_a}(\bar{\Omega}),$$

$$d\mu_{3\ a} = i_{V_a}(J).$$

Here $\Omega = \frac{1}{2}\Omega_{IJ} d\phi^I \wedge d\phi^J$ is the holomorphic symplectic form, $J = i g_{I\bar{K}} d\phi^I \wedge d\phi^{\bar{K}}$ the Kähler form on X, and $i_V(\omega)$ stands for the inner product of a vector field V with a form ω. We will assume that the moment maps exist globally (this is automatic if X is simply-connected). The function μ_+ is holomorphic, while $\mu_- = \overline{\mu}_+$ is antiholomorphic. They satisfy

$$\{\mu_{+a}, \mu_{+b}\} = -f_{ab}^c \mu_{+c}.$$

where the curly brackets are the Poisson brackets with respect to the complex symplectic form Ω_{IJ}. Similar formulas hold for μ_- and μ_3, with appropriate symplectic forms.

Another ingredient we need is a G-invariant nondegenerate symmetric bilinear form on the Lie algebra $\mathfrak{g}$. We will denote it κ_{ab} and its inverse κ^{ab}. It satisfies

$$\kappa_{ad} f_{bc}^d + \kappa_{bd} f_{ac}^d = 0.$$

Later we will subject κ to an integrality constraint: its restriction to the cocharacter lattice of G (the lattice of homomorphisms from $U(1)$ to the maximal torus of G) must be integral.

Finally, we will require the moment map μ_+ to satisfy

$$\mu_+ \cdot \mu_+ = 0,$$

where $\mu_+ \cdot \mu_+ = \kappa^{ab}\mu_{+a}\mu_{+b}$. We will see that this is necessary for the BRST transformation to be nilpotent on gauge-invariant observables.

The fields of the theory are

$$\text{bosonic:} \quad \phi^I, \phi^{\bar{I}}, A^a_\mu \qquad \text{fermionic:} \quad \eta^{\bar{I}}, \chi^I_\mu. \tag{1}$$

where I, $\bar{I} = 1, \ldots, 2n$, $\mu = 1, 2, 3$, $a = 1, \ldots \dim G$. $A^a_\mu dx^\mu$ is a connection 1-form on a principal G-bundle $\mathcal{E}$ over M. With respect to an infinitesimal gauge transformation with a parameter $\epsilon^a(x)$ it transforms as follows:

$$\delta_\epsilon A^a = -\left(d\epsilon^a - f^a_{bc}A^b\epsilon^c\right) = -D\epsilon^a.$$

Since the group G acts on X, there is a fiber bundle over M associated with $\mathcal{E}$ and typical fiber X. Let us call it $X_{\mathcal{E}}$. The connection 1-form A defines a nonlinear connection on $X_{\mathcal{E}}$, which locally can be thought of as a 1-form on M with values in the Lie algebra of vector fields on X. Concretely, this 1-form is given by

$$A^a V_a.$$

Bosonic fields $\phi^I(x), \phi^{\bar{I}}(x)$ describe a section ϕ of $X_{\mathcal{E}}$. Their covariant differentials are defined by

$$(D\phi)^I = d\phi^I + A^a V^I_a, \quad (D\phi)^{\bar{I}} = d\phi^{\bar{I}} + A^a V^{\bar{I}}_a.$$

The fermionic fields χ^I_μ are components of a 1-form χ^I on M with values in $\phi^*(T_{X_{\mathcal{E}}})$, where $T_{X_{\mathcal{E}}}$ is the $(1,0)$ part of the fiberwise-tangent bundle of $X_{\mathcal{E}}$. The fermionic field $\eta^{\bar{I}}$ is a 0-form on M with values in the complex-conjugate bundle $\phi^*(\overline{T}_{X_{\mathcal{E}}})$.

BRST transformations of the fields are postulated to be

$$\begin{aligned}
\delta_Q\phi^{\bar{I}} &= \eta^{\bar{I}}, \\
\delta_Q\phi^I &= 0 \\
\delta_Q\eta^{\bar{I}} &= -\bar{\xi}^{\bar{I}}, \\
\delta_Q\chi^I &= D\phi^I \\
\delta_Q A^a &= \kappa^{ab}\chi^K\partial_K\mu_{+b}
\end{aligned} \tag{2}$$

where $D\phi^I = d\phi^I + A^a V^I_a$ and we define

$$\xi^I := V^I \cdot \mu_-, \quad \bar{\xi}^{\bar{I}} := V^{\bar{I}} \cdot \mu_+$$

Note that δ_Q^2 is a gauge transformation with a parameter $\epsilon^a = -\kappa^{ab}\mu_{+b}$:

$$\delta_Q^2 A^a = \kappa^{ab}\left(d\mu_{+b} + f_{cb}^d A^c \mu_{+d}\right)$$

$$\delta_Q^2 \phi^I = 0, \quad \delta_Q^2 \phi^{\bar{I}} = -V^{\bar{I}} \cdot \mu_+$$

$$\delta_Q^2 \chi^I = -\chi^J \partial_J V^I \cdot \mu_+, \quad \delta_Q^2 \eta^{\bar{I}} = -\eta^{\bar{J}} \partial_{\bar{J}} V^{\bar{I}} \cdot \mu_+.$$

To compute Q^2 we used $V_a^K \Omega_{KJ} V_b^J = f_{ab}^c \mu_{+c}$ and $V^I \cdot \mu_+ = 0$.

The BRST differential is odd with respect to the $\mathbb{Z}_2$-grading given by fermion number modulo 2. In general, it is not possible to promote this $\mathbb{Z}_2$-grading to a $\mathbb{Z}$-grading (i.e. to define a $\mathbb{Z}$-valued ghost number so that δ_Q has ghost number 1). However, if X has a $U(1)$ action which commutes with the G-action and with respect to which Ω_{IJ} has charge 2, one can define a $U(1)$ ghost number symmetry as follows: its action on $\phi^I, \phi^{\bar{I}}$ comes from the $U(1)$ action on X, while the ghost numbers of fields χ, A, η are $-1, 0, 1$, respectively. Taking into account that μ_+ has ghost number 2, it is easy to check that δ_Q has ghost number 1. This situation occurs when X is the cotangent bundle of a complex manifold Y and $U(1)$ acts multiplicatively on the fiber coordinates.

2.2 The classical action

The BRST invariant action[3] consists of three parts:

$$S = \frac{1}{\hbar}\int_M \mathcal{L}, \quad \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{CS},$$

where[4]

$$\mathcal{L}_{CS} = \frac{1}{2}\kappa_{ab}\Big(A^a \wedge dA^b - \frac{1}{3} f_{cd}^b A^a \wedge A^c \wedge A^d\Big)$$

$$\mathcal{L}_1 = \delta_Q\Big(g_{I\bar{K}}\chi^I \wedge *D\phi^{\bar{K}} - \sqrt{h} g_{I\bar{K}}\xi^I \eta^{\bar{K}}\Big) = g_{I\bar{K}}\Big(D\phi^I \wedge *D\phi^{\bar{K}} - \chi^I \wedge \star D\eta^{\bar{K}}\Big) +$$

$$\sqrt{h}\Big(g_{I\bar{K}}\xi^I \bar{\xi}^{\bar{K}} + g_{I\bar{K}}\partial_{\bar{P}}(\xi^I)\eta^{\bar{K}}\eta^{\bar{P}}\Big)$$

$$\mathcal{L}_2 = \frac{1}{2}\Omega_{IJ}\left(\chi^I \wedge D\chi^J + \frac{1}{3}\mathcal{R}^J_{KL\bar{M}}\chi^I \wedge \chi^K \wedge \chi^L \wedge \eta^{\bar{M}}\right).$$

Here star denotes the Hodge star operator on forms on M with respect to a Riemannian metric $h_{\mu\nu}$, and covariant derivatives are defined as:

$$D\phi^I = d\phi^I + A \cdot V^I, \quad D\chi^I = \nabla\chi^I + A \cdot \nabla_K(V^I)\chi^K, \quad D\eta^{\bar{I}} = \nabla\eta^{\bar{I}} + A \cdot \nabla_{\bar{K}}(V^{\bar{I}})\eta^{\bar{K}}$$

[3] We use Euclidean conventions for the path-integral $Z = \int e^{-S}$.

[4] The overall normalization of the Q-exact piece is chosen for convenience.

where ∇ involves the Levi-Civita connection on X, i.e.

$$\nabla \eta^{\bar{J}} = d\eta^{\bar{J}} + \Gamma^{\bar{J}}_{\bar{I}\bar{K}} d\phi^{\bar{I}} \wedge \eta^{\bar{K}}, \quad \nabla \chi^J = d\chi^J + \Gamma^J_{IK} d\phi^I \wedge \chi^K.$$

(From now on, we will omit the sign $\wedge$ when writing the exterior product of forms on M.) Finally, $\mathcal{R}^J_{KL\bar{M}}$ denotes the curvature tensor of the Levi-Civita connection on X:

$$\mathcal{R}^J_{KL\bar{M}} = \frac{\partial \Gamma^J_{KL}}{\partial \phi^{\bar{M}}}, \quad \Gamma^I_{JK} = \left(\partial_J g_{K\bar{M}}\right) g^{I\bar{M}}.$$

Gauge-invariance of the Chern-Simons action with respect to large gauge transformations imposes a quantization condition on the symmetric form $\kappa_{ab}/\hbar$. If κ is chosen to be an integral pairing on the cocharacter lattice of G, then the quantization condition says

$$\hbar = \frac{2\pi i}{k}, \quad k \in \mathbb{Z}.$$

The classical limit is $k \to \infty$.

When checking the BRST-invariance of the action the following two identities are useful:

$$(\nabla_K \nabla_L \partial_I \mu_+) \cdot \mu_+ + 3\partial_I \mu_+ \cdot (\nabla_K \partial_L \mu_+) = 0 \quad (IKL) \tag{3}$$

$$\partial_{\bar{M}} \nabla_I \partial_K \mu_{3\ a} = 0 \tag{4}$$

where (IKL) in (3) indicates symmetrization in indices I, K, L. The first one follows from differentiating $\mu_+^2 = 0$ and using that Ω_{IJ} is covariantly constant with respect to the Levi-Civita connection. The second one follows from the definition of $\mu_{3\ a}$ and $\partial_I V^{\bar{J}}_a = 0$.

The most non-trivial step in checking the BRST invariance of the action is the cancelation of the two terms O_1 and O_2 arising from $\delta_Q \mathcal{L}_2$:

$$O_1 := -\frac{1}{2}\Omega_{IJ}\chi^I(\nabla_K V^J) \cdot (\delta_Q A)\chi^K$$

is canceled by

$$O_2 := -\frac{1}{6}\Omega_{IJ}\mathcal{R}^J_{KL\bar{M}}\chi^I\chi^K\chi^L(\delta_Q \eta^{\bar{M}})$$

To see this we first rewrite O_1 as

$$-\frac{1}{2}\chi^I\chi^K\chi^L\left(\nabla_K \partial_I \mu_+\right) \cdot \partial_L \mu_+ = \frac{1}{6}\chi^I\chi^K\chi^L\left(\nabla_K \nabla_L \partial_I \mu_+\right) \cdot \mu_+$$

where we used (3). Then we further rewrite O_1 as

$$\frac{1}{6}\chi^I\chi^K\chi^L\Omega_{IJ}\left(\nabla_K \nabla_L V^J\right) \cdot \mu_+ = -\frac{1}{6}\chi^I\chi^K\chi^L\Omega_{IJ}g^{J\bar{M}}\left(\nabla_K \nabla_L \partial_{\bar{M}} \mu_3\right) \cdot \mu_+$$

Now we use

$$\nabla_K \nabla_L \partial_{\bar{M}} \mu_{3\ a} = \partial_{\bar{M}} \nabla_K \partial_L \mu_{3\ a} + \mathcal{R}^P_{KL\bar{M}} \partial_P \mu_{3\ a}$$

and (4) to bring O_1 to the form

$$O_1 = -\frac{1}{6}\chi^I \chi^K \chi^L \Omega_{IJ} g^{J\bar{M}} \mathcal{R}^P_{KL\bar{M}} \partial_P \mu_3 \cdot \mu_+ = -\frac{1}{6}\chi^I \chi^K \chi^L \Omega_{IJ} g^{J\bar{M}} \mathcal{R}^P_{KL\bar{M}} g_{P\bar{N}} V^{\bar{N}} \cdot \mu_+$$

Lastly we observe that for Kähler manifolds

$$\mathcal{R}_{\bar{N}KL\bar{M}} := \mathcal{R}^P_{KL\bar{M}} g_{P\bar{N}} = \partial_{\bar{M}} \partial_{\bar{N}} \partial_K \partial_L \mathcal{K} - \Gamma^P_{KL} \partial_P \partial_{\bar{M}} \partial_{\bar{N}} \mathcal{K}$$

i.e. $\mathcal{R}_{\bar{N}KL\bar{M}}$ is symmetric in $\bar{N}, \bar{M}$ indices. In this way we obtain

$$O_1 = -\frac{1}{6}\chi^I \chi^K \chi^L \Omega_{IJ} \mathcal{R}^J_{KL\bar{N}} V^{\bar{N}} \cdot \mu_+$$

and we conclude using $\delta_Q \eta^{\bar{J}}$ that

$$O_1 + O_2 = 0.$$

2.3 Gauge-fixing

Next we discuss gauge-fixing in the CSRW model. This is somewhat nontrivial, because even before gauge-fixing we have a BRST operator δ_Q. When we extend the theory by adding Faddeev-Popov ghosts, anti-ghosts and Lagrange multiplier fields, we have to define how δ_Q acts on them. The possibilities are actually quite limited, since the total BRST operator

$$\delta_{\hat{Q}} = \delta_Q + \delta_{FP},$$

must be nilpotent. Here δ_{FP} is the usual Faddeev-Popov BRST operator. Note that the original BRST operator δ_Q is nilpotent only up to a gauge transformation.

We introduce fermionic Faddeev-Popov ghost and anti-ghost fields $c^a, \bar{c}_a$, as well as bosonic Lagrange multiplier fields B_a. We regard c as taking values in $\mathfrak{g}$ and $\bar{c}$ and B as taking values in $\mathfrak{g}^*$. The modified BRST operator $\hat{Q}$ acts as

$$\delta_{\hat{Q}} A_a = dc_a - f_{abd} A^b c^d + \chi^K \partial_K \mu_{+a}$$

$$\delta_{\hat{Q}} \phi^I = -V^I \cdot c, \quad \delta_{\hat{Q}} \phi^{\bar{I}} = \eta^{\bar{I}} - V^{\bar{I}} \cdot c$$

$$\delta_{\hat{Q}} \chi^I = D\phi^I + (\partial_J V^{Ia}) \chi^J c_a$$

$$\delta_{\hat{Q}} \eta^{\bar{I}} = -\bar{\xi}^{\bar{I}} + (\partial_{\bar{J}} V^{\bar{I}a}) \eta^{\bar{J}} c_a$$

$$\delta_{\hat{Q}} c^a = \kappa^{ab} \mu_{+b} - \frac{1}{2} f^a_{bc} c^b c^c, \quad \delta_{\hat{Q}} \bar{c} = B, \quad \delta_{\hat{Q}} B = 0.$$

It is easy to check that $\delta_{\hat{Q}}^2 = 0$.

One can express this result by saying that B and $\bar{c}$ are invariant under δ_Q, while the ghost field is not:

$$\delta_Q c^a = \kappa^{ab}\mu_{+b}.$$

The action of δ_{FP} on all fields is standard.

Let f_a be a gauge-fixing function, then a $\delta_{\hat{Q}}$-invariant gauge-fixed action has the form

$$S = \frac{1}{\hbar}\int_M \mathcal{L}, \quad \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{CS},$$

$$\mathcal{L}_{CS} = \frac{1}{2}\Big(\kappa_{ab}A^a \wedge dA^b - \frac{1}{3}f_{abc}A^a \wedge A^b \wedge A^c\Big)$$

$$\mathcal{L}_1 = \delta_{\hat{Q}}\Big(g_{I\bar{K}}\chi^I \wedge *D\phi^{\bar{K}} - \sqrt{h}g_{I\bar{K}}\xi^I\eta^{\bar{K}} + \bar{c}_a f^a\Big) = g_{I\bar{K}}\Big(D\phi^I \wedge *D\phi^{\bar{K}} - \chi^I \wedge \star D\eta^{\bar{K}}\Big) +$$
$$\sqrt{h}\Big(g_{I\bar{K}}\xi^I\bar{\xi}^{\bar{K}} + g_{I\bar{K}}\partial_{\bar{P}}(\xi^I)\eta^{\bar{K}}\eta^{\bar{P}}\Big) + B_a f^a - \bar{c}_a\delta_{\hat{Q}}f^a$$

$$\mathcal{L}_2 = \frac{1}{2}\Omega_{IJ}\left(\chi^I \wedge D\chi^J + \frac{1}{3}\mathcal{R}^J_{KL\bar{M}}\chi^I \wedge \chi^K \wedge \chi^L \wedge \eta^{\bar{M}}\right).$$

Note that the part of the action involving ghosts and anti-ghosts is not standard, since it involves the $\delta_{\hat{Q}}$-variation of the gauge-fixing function rather than the usual δ_{FP} variation. For example, if f^a depends only on the gauge field (e.g. one could pick the Lorenz gauge $f^a = \partial^\mu A^a_\mu$), the action contains a term which couples $\bar{c}^a$ to the "matter" fermion χ^K.

3 CSRW model for a flat target space

3.1 Relation to supergroup Chern-Simons theory

It was shown in [1] that constraints of $N = 4$ superconformal symmetry amount to the following quadratic constraints on the moment maps:

$$\mu_+ \cdot \mu_+ = 0, \quad \mu_3 \cdot \mu_+ = 0, \quad 2\mu_3 \cdot \mu_3 - \mu_+ \cdot \mu_- = 0. \tag{5}$$

Equivalently, if we define a 3-vector of moment maps $\mu_{i\,a} = (\mu_{1a}, \mu_{2a}, \mu_{3a})$ by letting

$$\mu_+ = \mu_1 + i\mu_2, \quad \mu_- = \mu_1 - i\mu_2,$$

then the constraint says that the traceless part of the symmetric tensor

$$K_{ij} = \kappa^{ab}\mu_{i\,a}\mu_{i\,b}$$

vanishes.

In the case when X is a vector space with a linear action of G the functions μ_{ia} are quadratic. For example, $\mu_{+a} = \frac{1}{2}\kappa_{ab}\tau^b_{IJ}\phi^I\phi^J$, where τ^b_{IJ} are constants. It was noted in [1] that the quadratic constraints on μ_{ia} are equivalent to the requirement that τ_{aIJ} together with f^a_{bc} are structure constants of a Lie superalgebra whose even part is $\mathfrak{g}$ and odd part is X.

This connection of $N=4$ $d=3$ superconformal field theories with Lie superalgebras seems quite mysterious. In this section we demystify it to some extent. We show that for flat X the topologically twisted Gaiotto-Witten theory is the supergroup Chern-Simons theory with a partial gauge-fixing (the odd part of the supergroup gauge-invariance is fixed, while the even one is not). The BRST differential δ_Q arises from such a partial gauge-fixing. Upon gauge-fixing the residual bosonic gauge symmetry, the twisted Gaiotto-Witten model becomes the supergroup Chern-Simons theory with the usual Faddeev-Popov gauge-fixing. The bosonic fields $\phi^I, \phi^{\bar{I}}$ are interpreted as bosonic Faddeev-Popov ghosts and anti-ghosts corresponding to odd gauge symmetries.

We begin by recalling that for flat X the moment maps take the form

$$\mu_{+a} = \frac{1}{2}\kappa_{ab}\tau^b_{IJ}\phi^I\phi^J, \quad \mu_{-a} = \frac{1}{2}\kappa_{ab}\tau^b_{\bar{I}\bar{J}}\phi^{\bar{I}}\phi^{\bar{J}}, \quad \mu_{3a} = -i\kappa_{ab}\phi^I\tau^b_{IJ}\Omega^{JK}g_{K\bar{M}}\phi^{\bar{M}}, \tag{6}$$

where $\tau^a_{\bar{I}\bar{L}} = \Omega_{\bar{I}\bar{P}}g^{\bar{P}M}\tau^a_{MJ}\Omega^{JK}g_{K\bar{L}}$ and $\Omega^{IJ} = -\frac{1}{4}g^{I\bar{K}}\Omega_{\bar{K}\bar{M}}g^{\bar{M}J}$ is the inverse of Ω_{IJ}.

Following [1], we introduce a Lie superalgebra $\mathfrak{G}$ whose even part is $\mathfrak{g}$ and odd part is the vector space X. Let M_a be a basis in $\mathfrak{g}$ and λ_J be a basis in X dual to coordinate functions ϕ^J. Then the commutation relations of $\mathfrak{G}$ are defined to be

$$[M_a, M_b] = f^c_{ab}M_c, \quad [M_a, \lambda_I] = \kappa_{ab}\tau^b_{IJ}\Omega^{JK}\lambda_K, \quad \{\lambda_I, \lambda_J\} = \tau^a_{IJ}M_a.$$

The super-Jacobi identities are equivalent to (5). Note also that $\mathfrak{G}$ has a natural super-trace (i.e. nondegenerate ad-invariant graded-symmetric bilinear form):

$$\mathrm{STr}(M_aM_b) = \kappa_{ab}, \quad \mathrm{STr}(\lambda_I\lambda_J) = \Omega_{IJ}.$$

This enables one construct a Chern-Simons action based on a supergroup associated to $\mathfrak{G}$. We will call it super-Chern-Simons theory.

It turns out that for flat X one can rewrite the action of the CSRW model as the action of the super-Chern-Simons theory with gauge-fixed odd part of the gauge symmetry. To see this, we introduce the following fields with values in $\mathfrak{G}$:

$$\mathcal{A} = \mathcal{A}_b + \mathcal{A}_f, \quad \mathcal{A}_b = A^aM_a, \quad \mathcal{A}_f = \chi^I\lambda_I, \tag{7}$$

$$\overline{C} = \phi^{\bar{I}} g_{\bar{I}K} \Omega^{KJ} \lambda_J, \quad C = \phi^I \lambda_I, \quad B = \eta^{\bar{M}} g_{\bar{M}K} \Omega^{KI} \lambda_I. \tag{8}$$

Here C and $\bar{C}$ are Faddeev-Popov ghosts and anti-ghosts for the fermionic gauge symmetry, while B is a fermionic Lagrange multiplier field.

The BRST operator δ_Q of the CSRW model can be interpreted as arising from gauge-fixing the fermionic part of the gauge symmetry. in super-Chern-Simons theory. In terms of the fields (7),(8) δ_Q acts as follows:

$$\delta_Q \mathcal{A} = dC - [\mathcal{A}, C\}, \quad \delta_Q \overline{C} = B, \quad \delta_Q C = 0,$$
$$\delta_Q B = \frac{1}{2}[\overline{C}, [C, C\}\}.$$

The symbol [} stands for the graded commutator in the Lie algebra.

The first three of these transformation laws are standard. (The BRST variation of C vanishes because we introduced ghosts only in the odd part of $\mathfrak{G}$). The transformation law of the Lagrange multiplier field B is unusual: in the Faddeev-Popov gauge-fixing procedure the BRST-variation of the Lagrange multiplier field is zero. This difference arises because the odd part of $\mathfrak{G}$ is not a Lie subalgebra. For this reason, δ_Q is not nilpotent when acting on A, but satisfies

$$\delta_Q^2 A^a = \kappa^{ab} D\mu_{+b}.$$

This is a gauge transformation with respect to the residual gauge symmetry with a parameter $-\kappa^{ab}\mu_{+b}$. Consistency requires that δ_Q^2 act as a gauge transformation on all fields, and this determines the BRST variation of B.

We note in passing that one can do partial gauge-fixing in ordinary Yang-Mills theory with purely bosonic gauge symmetry. For example, one can consider gauge-fixing the off-diagonal part of $SU(2)$ gauge symmetry in an $SU(2)$ Yang-Mills theory. The corresponding BRST operator will not be nilpotent because the part of the gauge symmetry that we fix is not a subalgebra of the full gauge symmetry. Rather, its square will be a gauge transformation with respect to residual $U(1)$ gauge symmetry. Of course, once we gauge-fix the residual $U(1)$ symmetry, we recover the usual theory with a nilpotent BRST operator.

In terms of $\mathfrak{G}$-valued fields the action of the CSRW model takes the form

$$S = \hbar^{-1} \int_M \mathcal{L}, \quad \mathcal{L} = \frac{1}{2}\mathrm{STr}\Big(\mathcal{A}\, d\mathcal{A} - \frac{1}{3}\mathcal{A}[\mathcal{A}, \mathcal{A}\}\Big) - \delta_Q \Psi \tag{9}$$

where the gauge-fixing fermion Ψ is taken to be

$$\Psi = \mathrm{STr}\Big(\mathcal{A}_f \star \big(d\overline{C} - [\mathcal{A}_b, \overline{C}\}\big) - 2\mathrm{vol}_M \mathrm{STr}\Big([\bar{C}, \bar{C}\}[C, B\}\Big).$$

This shows that for flat X the CSRW model is equivalent to super-Chern-Simons theory with the fermionic part of the gauge symmetry fixed.

As discussed in Section 2.3, we can gauge-fix the remaining bosonic gauge symmetry in the CSRW model by introducing the fermionic fields $c^a, \bar{c}_a$ and bosonic fields B_a and modifying the BRST operator δ_Q. In the case of flat X it is illuminating to introduce the fields taking values in $\mathfrak{G}$:

$$\mathbf{C} = c^a M_a + \phi^I \lambda_I, \quad \bar{\mathbf{C}} = \bar{c}^a M_a + \phi^{\bar{I}} g_{\bar{I}K} \Omega^{KJ} \lambda_J$$
$$\mathbf{B} = B^a M_a + (\eta^{\bar{I}} - V^{\bar{I}} \cdot c) g_{\bar{I}K} \Omega^{KJ} \lambda_J.$$

In terms of these fields the modified BRST differential $\delta_{\hat{Q}}$ acts as follows:

$$\delta_{\hat{Q}} \mathcal{A} = d\mathbf{C} - [\mathcal{A}, \mathbf{C}\}, \quad \delta_{\hat{Q}} \overline{\mathbf{C}} = \mathbf{B}, \quad \delta_{\hat{Q}} \mathbf{C} = -\frac{1}{2}[\mathbf{C}, \mathbf{C}\}, \quad \delta_{\hat{Q}} \mathbf{B} = 0.$$

These are the usual BRST transformations laws. The nilpotency of the operator $\delta_{\hat{Q}}$ follows immediately from the Jacobi identities for $\mathfrak{G}$. Note that the new Lagrange multiplier field $\mathbf{B}$ is BRST-invariant, as it should. In terms of the new fields the gauge-fixed action of the CSRW model is identical to the fully gauged-fixed super-Chern-Simons action for a particular choice of the gauge-fixing fermion:

$$S_{g.f.} = \hbar^{-1} \int_M \mathcal{L}, \quad \mathcal{L} = \frac{1}{2} \mathrm{STr}\Big(\mathcal{A}\, d\mathcal{A} - \frac{1}{3} \mathcal{A} [\mathcal{A}, \mathcal{A}\} \Big) - \delta_{\hat{Q}} \hat{\Psi} \tag{10}$$

where

$$\hat{\Psi} = \mathrm{STr}\Big(\mathcal{A} \star \big(d\overline{\mathbf{C}} - [\mathcal{A}, \overline{\mathbf{C}}\} \big) \Big) - 2\mathrm{vol}_M \mathrm{STr}\Bigg([\bar{\mathbf{C}}, \bar{\mathbf{C}}\} [\mathbf{C}, \Big(\mathbf{B} + [\overline{\mathbf{C}}, \mathbf{C}\} \Big) \} \Bigg) \tag{11}$$

The only unusual thing about it is the form of the gauge-fixing fermion (usually it is taken to be independent of ghost and anti-ghost fields). These ghost-dependent terms are introduced to reproduce the scalar potential $V = \xi^I g_{I\bar{J}} \bar{\xi}^{\bar{J}}$ and the fermionic mass term $g_{I\bar{K}} \partial_{\bar{J}} (\xi^I) \eta^{\bar{K}} \eta^{\bar{J}}$ present in the CSRW model.

3.2 Local observables

To find topological observables, we have to compute the cohomology of the BRST operator $\hat{Q}$ in the space of 0-forms. In Chern-Simons theory with a compact gauge group one usually restricts to observables with ghost number 0. Then the only observable is the identity operator, basically because there is nothing to construct the local observable from other than the Faddeev-Popov ghosts. If we impose the same restriction in the super-Chern-Simons theory, we get the same trivial result, for exactly the same reason.

However, if we regard the CSRW model as a gauged version of the Rozansky-Witten model, it seems unreasonable to restrict oneself to observables of ghost number 0. In ordinary Chern-Simons theory the ghost-number zero sector is distinguished by its unitarity properties, but this is no longer the case in the CSRW model.

Candidate observables in the super-Chern-Simons theory are polynomial functions of the $\mathfrak{G}$-valued field $\mathbf{C}$. Its BRST transformation is

$$\delta_{\hat{Q}}\mathbf{C} = -\frac{1}{2}[\mathbf{C},\mathbf{C}\}.$$

This is simply the Chevalley-Eilenberg differential in the complex which computes the cohomology of $\mathfrak{G}$ with trivial coefficients.[5] For $\mathfrak{G} = \mathfrak{gl}(m|n)$ and $\mathfrak{G} = \mathfrak{osp}(m|n)$ this cohomology is finite-dimensional [6]. For example, for $\mathfrak{gl}(m|n)$ it is isomorphic to the cohomology of the bosonic Lie algebra $\mathfrak{gl}(\max(m,n))$ [6]. In particular, the cohomology of $\mathfrak{gl}(1|1)$ is isomorphic to the exterior algebra with one generator of ghost number one.

From the point of view of the Gaiotto-Witten theory, this result may seem somewhat surprising, since an arbitrary G-invariant holomorphic function of ϕ^I is obviously $\delta_{\hat{Q}}$-closed. However, because of nonstandard transformation properties of the c^a ghosts, all such observables are $\delta_{\hat{Q}}$-exact. For example, for $\mathfrak{g} = \mathfrak{gl}(1|1)$ ($G = U(1)\times U(1)$) we have two complex scalars A, B with $U(1)\times U(1)$ charges $(1,-1)$ and $(-1,1)$, so the only gauge-invariant holomorphic function is AB. But this is proportional to $\delta_{\hat{Q}}(c_1+c_2)$, where c_1 and c_2 are the Faddeev-Popov ghosts for the two $U(1)$ factors.

Apart from ordinary local observables built as polynomials in the fields, there may also be local observables which are disorder operators. These are monopole operators, i.e. operators which insert a Dirac monopole singularity at a point [7, 8]. Here we limit ourselves to the simplest case $G = U(1)\times U(1)$, $\mathfrak{G} = \mathfrak{gl}(1|1)$. This case is simple because the bosonic part of $\mathfrak{G}$ is abelian. The monopole sits in the bosonic part of the gauge group and is characterized by the property that the gauge field strengths F_1 and F_2 are singular at the insertion point:

$$F_1 = *d\frac{m_1}{2r} + \text{regular}, \quad F_2 = *d\frac{m_2}{2r} + \text{regular},$$

where r is the distance to the insertion point and m_1, m_2 are integers (magnetic charges). We will denote the corresponding operator M_{m_1,m_2}. In the presence of such a singularity the Chern-Simons action is not gauge-invariant:

$$\delta_\epsilon S_{CS} = ik\left(m_1\epsilon_1 - m_2\epsilon_2\right),$$

[5] Similarly, if we allowed ghost-dependent observables in the ordinary Chern-Simons theory with gauge group G, we would get the cohomology of $\mathfrak{g}$ with trivial coefficients as the answer.

where ϵ_1, ϵ_2 are the parameters of the $U(1) \times U(1)$ gauge transformation. Thus the monopole operator has electric charges $km_1, -km_2$. Upon gauge-fixing, this translates into the following transformation law of M_{m_1,m_2} under the BRST-transformation:

$$\delta_{\hat{Q}} M_{m_1,m_2} = -ik\,(m_1 c_1 - m_2 c_2)\, M_{m_1,m_2}.$$

To get a BRST-invariant operator, one must multiply M_{m_1,m_2} by a suitable polynomial in the ghost fields (both bosonic and fermionic). Since $\delta_{\hat{Q}}$ variations of all these fields do not involve c_1+c_2, it is obvious that a necessary condition for the existence of a suitable polynomial is $m_1 = m_2 = m$. Then for $km > 0$ the BRST-invariant combination has the form

$$f(A, B, c_1, c_2) B^{km} M_{m_1,m_2},$$

while for $km < 0$ it has the form

$$f(A, B, c_1, c_2) A^{-km} M_{m_1,m_2}.$$

Here $f(A, B, c_1, c_2)$ is an arbitrary BRST-invariant function of A, B, c_1, c_2. We also must identify functions which differ by BRST-exact terms. This is exactly the same problem as what we encountered above when computing the space of "ordinary" local observables. As discussed above, this space is isomorphic to the cohomology of the bosonic Lie algebra $\mathfrak{gl}(1)$, which is an exterior algebra with one generator with ghost number one. This generator is $c_1 + c_2$. Thus for every nonzero magnetic charge the space of BRST-invariant monopole operators is two-dimensional and has a one-dimensional even subspace and one-dimensional odd subspace. The even component has ghost number $|km|$, while the odd component has ghost number $|km| + 1$.

For $m = 0$ the situation is similar: $\delta_{\hat{Q}}$-cohomology is spanned by 1 and c_1+c_2. As mentioned above, this is interpreted as the cohomology of $\mathfrak{G}$ with trivial coefficients.

To summarize, the space of local observables in the CSRW model with gauge group $U(1) \times U(1)$ at level k is the tensor product of the cohomology of $\mathfrak{G}$ (which is isomorphic to the exterior algebra with one generator of ghost number 1) and an infinite-dimensional vector space V graded by the magnetic charge $m \in \mathbb{Z}$. We will refer to the two factors in this tensor product as perturbative and nonperturbative spaces. For any m the component V_m of the nonperturbative space is one-dimensional and has ghost number $|km|$.

The space of local observables in a 3d TFT must be an associative graded-commutative algebra. For $G = U(1) \times U(1)$ it is easy to determine the algebra structure. First of all, it is clear that the factorization into perturbative and nonperturbative parts persists on the algebra level. The perturbative algebra is the exterior algebra with one generator. The nonperturbative algebra is tightly constrained by the conservation of magnetic charge: the product of two monopole operators with magnetic charges m, n is a monopole operator with magnetic charge $m+n$. Further, since AB is $\delta_{\hat{Q}}$-exact, the product of monopole operators with $mn < 0$ is $\delta_{\hat{Q}}$-exact. Thus the nonperturbative algebra is a commutative algebra with three generators $1, x, y$ and relations

$$1 \cdot x = x, \quad 1 \cdot y = y, \quad x \cdot y = 0.$$

Both generators x, y have ghost number $|k|$, while their magnetic charges are ± 1.

4 CSRW model for a curved target space

In this section we consider the CSRW model for a curved target space. We consider in detail the case $G = SU(2)$, $X = T^*\mathbb{CP}^1$. Then we discuss a generalization where G is an arbitrary compact simple Lie group and X is the cotangent bundle of the flag manifold $G_{\mathbb{C}}/B$.

4.1 Moment maps

The cotangent bundle of any complex manifold is a complex symplectic manifold. If x^i are local coordinates on the base and p_i are dual coordinate on the fibers, then the symplectic form is simply

$$\Omega = dp_i dx^i.$$

If the base manifold admits a holomorphic $G_{\mathbb{C}}$ action, then $G_{\mathbb{C}}$ acts on the total space of the cotangent bundle and this action is Hamiltonian. The corresponding complex moment maps are simply

$$\mu_{+a} = v_a^i p_i$$

where $v_a = v_a^i \partial_i$ is the holomorphic vector field on the base corresponding to a basis vector $e_a \in \mathfrak{g}_{\mathbb{C}}$. It is more difficult to satisfy the condition $\kappa^{ab}\mu_{+a}\mu_{+b} = 0$: this requires the vector fields v_a to be null with respect to the metric κ.

One simple situation where this happens is when the base manifold is itself the quotient of $G_{\mathbb{C}}$ by a Borel subgroup, i.e. the flag manifold of G. The simplest nontrivial case is $G = SU(2)$, in which case the flag manifold is $\mathbb{CP}^1$. It is well-known that $T^*\mathbb{CP}^1$ admits a hyper-Kähler metric known as the Eguchi-Hanson metric (see e.g. [9]). The group $SU(2)$ acts on it by isometries and preserves all three complex structures. Below we summarize some properties of this manifold and of the $SU(2)$ action on it.

Let z be an inhomogeneous coordinate on $\mathbb{CP}^1$ and b be a complex coordinate on the fiber of $T^*\mathbb{CP}^1$. The complexification of $SU(2)$ is $SL(2,\mathbb{C})$ and it acts as follows:

$$z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}, \quad b \mapsto b(\gamma z + \delta)^2 \tag{12}$$

Let us introduce $(0,1)$ forms

$$e_1 = \frac{1}{2}\frac{db}{b} + \bar{z}e_2, \quad e_2 = \frac{dz}{1+|z|^2}$$

such that e_1 and $e_2\sqrt{\frac{b}{\bar{b}}}$ are invariant under $SU(2)$. The Kähler form on $T^*\mathbb{CP}^1$ which respects the $SU(2)$ symmetry is

$$J = t\hat{J}, \quad \hat{J} = i\Big(f_1(x)e_1 \wedge \bar{e}_1 + f_2(x)e_2 \wedge \bar{e}_2\Big). \tag{13}$$

Here t sets the scale, $\int_{\mathbf{P}^1} \hat{J} = 2\pi$ and

$$f_1(x) = \frac{x^2}{f_2}, \quad f_2(x) = \sqrt{1+x^2} \tag{14}$$

are functions of the only $SU(2)$ invariant

$$x^2 = |b|^2(1+|z|^2)^2.$$

A harmonic holomorphic $SU(2)$-invariant (0,2) form Ω which respects $SU(2)$ symmetry is given by

$$\Omega = t\hat{\Omega}, \quad \hat{\Omega} = db \wedge dz \tag{15}$$

Note that theory does not dependent on t since all t-dependence in the BRST-non-exact piece of the action can be absorbed into rescaling χ and η. We note as an aside that the relation between our coordinates z and b and the standard coordinates r, θ, ϕ, ψ on the Eguchi-Hanson space [9] is

$$x = r^2(1-r^{-4})^{\frac{1}{2}}, \quad z = \frac{sin\theta}{1-cos\theta}e^{i\phi}, \quad b = \frac{xe^{i(\phi-\psi)}}{1+|z|^2}$$

From the transformation (12) we find the vector fields

$$V^z = (-i)\begin{pmatrix} -z & 1 \\ -z^2 & z \end{pmatrix}, \quad V^b = (-i)b\begin{pmatrix} 1 & 0 \\ 2z & -1 \end{pmatrix} \tag{16}$$

$$V^{\bar{z}} = i\begin{pmatrix} -\bar{z} & -\bar{z}^2 \\ 1 & \bar{z} \end{pmatrix}, \quad V^{\bar{b}} = i\bar{b}\begin{pmatrix} 1 & 2\bar{z} \\ 0 & -1 \end{pmatrix}$$

Using (13) and (15) we find the moment maps $\mu = t\hat{\mu}$ with

$$\hat{\mu}_+ = (-i)b \begin{pmatrix} -z & 1 \\ -z^2 & z \end{pmatrix}, \quad \hat{\mu}_- = i\bar{b} \begin{pmatrix} -\bar{z} & -\bar{z}^2 \\ 1 & \bar{z} \end{pmatrix} \tag{17}$$

$$\hat{\mu}_3 = \frac{f_2(x)}{1+|z|^2} \begin{pmatrix} \frac{1}{2}(1-|z|^2) & \bar{z} \\ z & -\frac{1}{2}(1-|z|^2) \end{pmatrix} \tag{18}$$

For $G = SU(2)$ the metric on the Lie algebra is given in terms of the trace Tr so that moment maps satisfy:

$$\mathrm{Tr}\Big(\mu_+^2\Big) = 0, \quad \mathrm{Tr}\Big(\mu_+\mu_3\Big) = 0, \quad 2\mathrm{Tr}\Big(\mu_3^2\Big) - \mathrm{Tr}\Big(\mu_+\mu_-\Big) = t^2. \tag{19}$$

4.2 Local observables

In this section we compute the algebra of local observables. We will see that it is two-dimensional and isomorphic to the cohomology of the Lie algebra $\mathfrak{sl}(2)$.

We begin with local observables which do not depend on the fermionic ghosts c^a. Such observables can be regarded as $(0,p)$ forms on the target space. Such a form ω is annihilated by $\delta_{\hat{Q}}$ if an only if it is $SU(2)$-invariant and is annihilated by

$$Q = \overline{\partial} + i_{\bar{\xi}}$$

and the where

$$\bar{\xi} = \frac{x^2}{\bar{b}}\partial_{\bar{z}} - \frac{2x^2 z}{1+|z|^2}\partial_{\bar{b}}.$$

Suppose first that ω is odd, i.e. it is a $(0,1)$ form. The requirement of $SU(2)$-invariance restricts its form to be

$$\omega_{odd} = C_1(x)\bar{e}_1 + C_2(x)\bar{e}_2\sqrt{\frac{\bar{b}}{b}}.$$

We find that $Q\omega_{odd} = 0$ implies $C_2(x) = 0$ and does not restrict $C_1(x)$. Note that $C_1(x)$ should be an even function of x and vanish at least as x^2 for $x \mapsto 0$ so that ω_{odd} is smooth. However, all such ω_{odd} are Q-exact since

$$C_1(x)\bar{e}_1 = \overline{\partial}F(x), \quad C_1(x) = xF'(x)$$

with smooth $F(x)$, vanishing at least as x^2 for $x \mapsto 0$.

Consider now an even form

$$\omega_{even} = A(x) + B(x)d\bar{b} \wedge d\bar{z}.$$

Here $A(x)$ and $B(x)$ are even functions of x which are smooth at $x \mapsto 0$ so that ω_{even} is also smooth. We find that $Q\omega_{even} = 0$ implies $A'(x) = 2xB(x)$. However,

$$\omega_{even} = Q(\Upsilon), \quad \Upsilon = c(x)\bar{e}_2\sqrt{\frac{\bar{b}}{b}}$$

where

$$A(x) = xc(x), \quad B(x) = \frac{1}{2}(c'(x) + \frac{c(x)}{x})$$

Υ is well behaved if and only if $c(x)$ is an odd function of x and $c(x) \mapsto x^{1+2m}$, $m \geq 0$ for $x \mapsto 0$. Therefore only the solution with $A(x) = 1$ is not BRST-trivial. We conclude that the BRST cohomology is one-dimensional corresponding to $A(x) = 1$, $B(x) = 0$.

If the fermionic ghosts transformed in the standard way, we could conclude from here that apart from 1 the only BRST cohomology classes are the ones constructed from the fermionic ghosts c^a alone. That is, the cohomology is isomorphic to the cohomology of the Lie algebra $\mathfrak{sl}(2)$. This cohomology has a single generator in degree three

$$\epsilon_{abc}c^a c^b c^c.$$

In our case, the BRST transformation of c^a is nonstandard, but this does not affect the structure of the cohomology. Indeed, we can view the difference between $\delta_{\hat{Q}}(c^a)$ and the ordinary Chevalley-Eilenberg differential acting on c^a as a perturbation and write down a spectral sequence which converges to the desired answer and whose E_1 term is the cohomology of $\mathfrak{sl}(2)$. But since the only nonzero terms in the E_1 term are of ghost number 0 and 3, there can be no nontrivial differentials and the spectral sequences collapses at the very first stage.[6]

Unlike the case of flat X, there are no BRST-invariant monopole operators in this model. To see this, let us adopt the radial quantization viewpoint, i.e. let us consider the space of states of theory on a manifold of the form $S^2 \times \mathbb{R}$ where $\mathbb{R}$ is regarded as time. Monopole singularity corresponds to a constant magnetic flux on S^2. This flux breaks the gauge group down to $U(1)$. Thanks to the Chern-Simons term the monopole operator has electric charge with respect to this unbroken $U(1)$, and this charge must be canceled by zero modes of other fields. However, there are no such zero modes. The bosonic field corresponding to the target coordinate b is massive, while the field z gets an effective potential from its interaction with a constant backround magnetic flux (the zeroes of this potential are the two points on $\mathbb{CP}^1$ which are fixed by the unbroken $U(1)$). The fermionic fields $\eta^{\bar{I}}$ are also massive. Hence no BRST-invariant monopole operators are possible.

[6] The concrete form of the nontrivial class of degree 3 is of course affected by the perturbation: it is easy to check that the $\hat{Q}$-closed combination is

$$\frac{1}{3}\epsilon_{abc}c^a c^b c^c - \mu_{+a}c^a$$

.

4.3 Generalization to an arbitrary gauge group

The discussion can be easily generalized to any compact simple Lie group G and $X = T^*(G_{\mathbb{C}}/B) = T^*(G/T)$ where T is a maximal torus of G. Let us denote $r = \dim\ T$ and expand the gauge field in the Cartan-Weyl basis of $\mathfrak{g}$:

$$A = A^i H_i + \sum_{\alpha\in\mathcal{S}} \left(A^\alpha E_\alpha + A^{\bar\alpha} E_{\bar\alpha}\right)$$

where $\mathcal{S}$ is the set of positive roots. Let w^α be complex coordinates on the coset G/T near the identity element of G:

$$\rho = e^{i\sum_{\alpha\in\mathcal{S}}\left(w^\alpha E_\alpha + \bar w^{\bar\alpha} E_{\bar\alpha}\right)} h, \quad h \in T$$

It is convenient to work with holomorphic Darboux coordinates on X, $\phi^I = (b_\alpha, z^\alpha)$ for $\alpha \in \mathcal{S}$, so that the holomorphic symplectic form is

$$\Omega = t \sum_{\alpha\in\mathcal{S}} db_\alpha \wedge dz^\alpha.$$

Near the unit of G, for $|w_\alpha| \ll 1$, we find $z^\alpha = w^\alpha + O(|w|^3)$. Similar to the previous discussion for $T^*\mathbb{CP}^1$ we introduced an overall scale factor t which enters both the holomorphic symplectic form $\Omega = t\hat\Omega$ and the Kähler form $J = t\hat J$.

The group G acts on the right coset from the left:

$$g : \rho \mapsto g\rho, \quad g \in G.$$

As explained above, the holomorphic moment map is linear in the fiber coordinates b_α:

$$\mu_+ = t\hat\mu_+, \quad \hat\mu_+ = \sum_{\alpha\in\mathcal{S}} b_\alpha V^{z^\alpha}(z) \tag{20}$$

where

$$V^{z^\alpha} = E_\alpha + iz^\alpha \sum_{j=1}^{r} \alpha^j H_j + \sum_\delta{}' z^{\alpha-\delta} E_\delta + O(z^2) \tag{21}$$

Note that G/T is a homogenous space so it is sufficient to work near the identity element of G, i.e. for $|z| \ll 1$. In (21) $\sum_\delta'$ means that we sum over $\delta \in \mathcal{S}$ such that $\alpha - \delta \in \mathcal{S}$.

From (20) we find

$$V^{b_\alpha} = \sum_{\beta\in\mathcal{S}} \partial_{z^\alpha} V^{z^\beta}, \quad \xi^{z^\alpha} = t\sum_\gamma N^{\alpha\bar\gamma} \bar b_{\bar\gamma}, \quad \xi^{b_\alpha} = t\sum_\gamma\sum_\beta \bar b_{\bar\gamma} b_\beta \partial_{z^\alpha} N^{\beta\bar\gamma}$$

where the matrix

$$N^{\alpha\bar\gamma} = \mathrm{Tr}\Big(V^{\bar z^{\bar\gamma}} V^{z_\alpha}\Big)$$

is non-degenerate. The nondegeneracy can be most easily seen at $z = 0$, and then must also be true in an open neighborhood of $z = 0$.

An important consequence of the nondegeneracy of the matrix N is that the fields b_α and the fermions $\eta^{\bar{I}}$ are massive. Thus the only zero modes are the bosons parameterizing the base $G_{\mathbb{C}}/B$. This implies that when the theory is considered on a manifold $S^2 \times \mathbb{R}$ (without magnetic flux) the BRST cohomology can be computed in the space of holomorphic functions on $G_{\mathbb{C}}/B$ tensored with the zero modes of the Faddeev-Popov ghosts. Since $G_{\mathbb{C}}/B$ is compact, the only holomorphic function is a constant, and we conclude that the BRST cohomology is isomorphic to the cohomology of the Lie algebra $\mathfrak{g}$ with trivial coefficients. Monopole operators do not arise for the same reason as for $G = SU(2)$.

5 Wilson loops in the CSRW model

5.1 General construction

In Chern-Simons theory the most important observables are the Wilson loops. Their correlators are known to give knot invariants associated with representation theory of quantum groups. Similarly, we expect correlators of Wilson loops in the CSRW model to compute some knot invariants. While it is very plausible that these invariants are also related in some way to quantum groups, the precise relationship is unclear at the moment.

In the Chern-Simons theory the Wilson loops are labeled by finite-dimensional irreducible representations of G.[7] In the the Rozansky-Witten model they are labeled by objects of the $\mathbb{Z}_2$-graded derived category of coherent sheaves on X. A natural guess is that Wilson loops for the CSRW model are labeled by objects of the $G_{\mathbb{C}}$-equivariant derived category of X. We will see in this section that this is almost correct: the relevant category is a certain interesting deformation of the $G_{\mathbb{C}}$-equivariant derived category of X. The existence of this deformation is tied to the fact that the $G_{\mathbb{C}}$-action is Hamiltonian and the moment map is isotropic.

Let $E = E^+ + E^-$ be a $\mathbb{Z}_2$−graded vector bundle over X. Its fiber over a point $p \in X$ is interpreted as the space of degrees of freedom living on the Wilson loop. Observables on the Wilson loop not involving ghosts can be regarded as sections of the graded algebra bundle $\mathrm{End}(E) \otimes \Omega^{0,\bullet}$. Bulk observables restricted to the Wilson loop take values in the subalgebra $\Omega^{0,\bullet}$. We need to define the action of Q on the observables on the Wilson loop so that its restriction

[7] This statement is true in the classical limit. For finite k it one wants to preserve unitarity one has to keep only the so-called integrable representations, i.e. those representations which can be deformed into irreducible representations of the quantum group.

to the subalgebra of bulk observables is given by (2). We try the following differential operator on $E \otimes \Omega^{0,\bullet}$:

$$D = \overline{\partial} + \mathcal{K} = \overline{\partial} + \begin{pmatrix} \omega^{+}_{\bar{I}} d\phi^{\bar{I}} & \mathcal{T} \\ \mathcal{S} & \omega^{-}_{\bar{I}} d\phi^{\bar{I}} \end{pmatrix},$$

where ω^+ and ω^- are connection $(0,1)$-forms on E^+ and E^-, $\mathcal{S} \in \mathrm{Hom}(E^+, E^-)$ and $\mathcal{T} \in \mathrm{Hom}(E^-, E^+)$. The operator D is a $\bar{\partial}$-superconnection on E. We will also need connection $(1,0)$-forms on E^+ and E^-

$$\partial^{\pm} = \partial + \omega^{\pm}_{I} d\phi^{I}.$$

The bulk BRST operator Q is not nilpotent: it squares to a gauge transformation with a parameter $\epsilon^a = \kappa^{ab}\mu_{+b}$. We should impose the same constraint on the BRST operator acting on the observables on the Wilson loop. In particular, this means that we need to have an action of $\mathfrak{g}$ on the space of smooth sections of E which is compatible with the action of $\mathfrak{g}$ on smooth functions on X. The latter is given by

$$e_a : f \mapsto V_a(f) = V^{\hat{P}} \partial_{\hat{P}} f, \quad e_a \in \mathfrak{g},\ f \in C^{\infty}(X).$$

Here hatted indices run both over holomorphic and anti-holomorphic values, e.g. $\hat{P} = (P, \bar{P})$. The general form of such an action is

$$e_a : s \mapsto \mathcal{V}_a(s) = V^{\hat{P}} \nabla_{\hat{P}} s + T_a s, \quad e_a \in \mathfrak{g},\ s \in \Gamma(E).$$

Here T_a is an even section of $\mathrm{End}(E)$ which we can write as a supermatrix:

$$T_a = \begin{pmatrix} t^{+}_{a} & 0 \\ 0 & t^{-}_{a} \end{pmatrix}$$

We will require the connection $\omega^{\pm}$ on $E^{\pm}$ to be $\mathfrak{g}$-invariant, i.e. $\mathcal{V}_a$ should map covariantly constant sections to covariantly constant sections. This gives

$$\nabla_{\hat{P}} t^{\pm}_{a} = V^{\hat{K}}_{a} \mathcal{F}^{\pm}_{\hat{K}\hat{P}}, \tag{22}$$

Here the curvature 2-forms for $E^{\pm}$ are defined by

$$\mathcal{F}^{\pm} = d\omega^{\pm} + \omega^{\pm} \wedge \omega^{\pm}.$$

The requirement that the operators $\mathcal{V}^a$ form a representation of the Lie algebra $\mathfrak{g}$ gives

$$[t^{\pm}_{a}, t^{\pm}_{b}] = f^{c}_{ab} t^{\pm}_{c} + V^{\hat{J}}_{a} V^{\hat{K}}_{b} \mathcal{F}^{\pm}_{\hat{J}\hat{K}} \tag{23}$$

We are going to impose[8] the following condition on the Wilson-loop BRST operator D:

$$D^2 = \kappa^{ab}\mu_{+a}T_b, \tag{24}$$

$$[D, \mathcal{V}_a] = 0. \tag{25}$$

The first condition essentially says that on holomorphic sections D^2 is a gauge transformation with a parameter $\kappa^{ab}\mu_{+b}$. Indeed, for holomorphic sections $\mathcal{V}_a$ reduces to

$$\mathcal{W}_a = V_a^I \nabla_I + T_a,$$

and since $\mu_+ \cdot \mu_+ = 0$, we have

$$\mu_+ \cdot \mathcal{W} = \mu_+ \cdot T.$$

Note that the differential operator $\mathcal{W}_a$ is holomorphic (commutes with $\nabla_{\bar{I}}$) thanks to the condition (22).

The second condition says that the BRST operator commutes with gauge transformations. It is understood there that $\mathcal{V}_a$ is lifted from E to $E \otimes \Omega^{0,\bullet}$. This lift is canonical: since $\mathfrak{g}$ acts on X, it also acts on forms on X via the Lie derivative.

These conditions are equivalent to the following constraints on the connections $\omega^\pm$ and the endomorphisms $\mathcal{S}$ and $\mathcal{T}$:

$$\mathcal{F}^{\pm}_{\bar{I}\bar{J}} = 0, \tag{26}$$

$$\nabla_{\bar{I}}\mathcal{S} = 0, \qquad \nabla_{\bar{I}}\mathcal{T} = 0, \tag{27}$$

$$\mathcal{S}\mathcal{T} = \kappa^{ab}\mu_{+a}t_b^-, \qquad \mathcal{T}\mathcal{S} = \kappa^{ab}\mu_{+a}t_b^+ \tag{28}$$

$$V_a^I\nabla_I\mathcal{T} = \mathcal{T}t_a^- - t_a^+\mathcal{T}, \qquad V_a^I\nabla_I\mathcal{S} = \mathcal{S}t_a^+ - t_a^-\mathcal{S}. \tag{29}$$

Here the covariant derivatives are given by

$$\nabla_{\hat{P}}\mathcal{T} = \partial_{\hat{P}}\mathcal{T} + \omega^+_{\hat{P}}\mathcal{T} - \mathcal{T}\omega^-_{\hat{P}}, \quad \nabla_{\hat{P}}\mathcal{S} = \partial_{\hat{P}}\mathcal{S} + \omega^-_{\hat{P}}\mathcal{S} - \mathcal{S}\omega^+_{\hat{P}}.$$

Note that the equations (26-29) imply, in particular, that $E^\pm$ are holomorphic vector bundles, and $\mathcal{S}$ and $\mathcal{T}$ are holomorphic bundle maps. However, we do not get a complex of vector bundles because $\mathcal{S}\mathcal{T}$ and $\mathcal{T}\mathcal{S}$ are not equal to zero, in general. This is similar to what happens in the category of matrix factorizations arising in the Landau-Ginzburg models [10, 11, 12].

Note also that the condition (25) can be simplified using (31). Namely, the part of $\mathcal{V}_a$ proportional to the anti-holomorphic component of V_a automatically commutes with D thanks to (26). Therefore (25) is really a condition on the holomorphic differential operator $\mathcal{W}_a = V_a^I\nabla_I + T_a$:

$$[D, \mathcal{W}_a] = 0.$$

[8]The idea of this construction arose in a conversation with Lev Rozansky.

Now let γ be a closed curve in M parameterized by $t \in [0,1)$. The fields $\phi^{\hat{I}}$ specify a section of the principal G-bundle $X_{\mathcal{E}}$ with fiber X. Our considerations are local, so we can choose a trivialization of $X_{\mathcal{E}}$ and think of $\phi^{\bar{I}}$ as a map from M to X. Independence of the choice of a trivialization will be ensured by keeping track of gauge-invariance.

We introduce a supermanifold $\Pi\bar{T}_X$ with odd coordinates $\eta^{\bar{I}}$. There is an obvious map $\pi : \Pi\bar{T}_X \mapsto X$, and we may regard $\mathcal{K}$ as a locally-defined odd section of the $\mathbb{Z}_2$-graded bundle $\pi^*\mathrm{End}(E)$. Similarly, $\overline{\partial}$ can be interpreted as an odd vector field $\eta^{\bar{I}}\partial_{\bar{I}}$ on $\Pi\bar{T}_X$, and D is a first-order differential operator on π^*E. Given a map $\Phi = (\phi, \eta) : \gamma \mapsto \Pi\bar{T}_X$ and a section $\chi_t^I dt$ of $\phi^*T_X \otimes T_\gamma^*$, we consider a connection 1-form on the pull-back of $\Phi^*(E)$:

$$\mathcal{N} = \begin{pmatrix} A_t^c t_c^+ - \omega_{\hat{I}}^+ \partial_t \phi^{\hat{I}} - \chi_t^N \eta^{\bar{M}} \mathcal{F}_{N\bar{M}}^+ & -\chi_t^I \nabla_I \mathcal{T} \\ -\chi_t^I \nabla_I \mathcal{S} & A_t^c t_c^- - \omega_{\hat{I}}^- \partial_t \phi^{\hat{I}} - \chi_t^N \eta^{\bar{M}} \mathcal{F}_{N\bar{M}}^- \end{pmatrix} dt. \tag{30}$$

Using (22) and (29), as well as $D^2 = \kappa^{ab}\mu_{+a}T_b$ we find

$$\delta_Q \mathcal{N} = -d_t(\mathcal{K}) + [\mathcal{N}, \mathcal{K}].$$

Hence one can construct a BRST-invariant Wilson loop by letting

$$\mathsf{W} = \mathrm{STr}\, \mathcal{U}(0,1)$$

where $\mathcal{U}(0,t)$ is the parallel transport operator of the connection $\mathcal{N}$, i.e. the unique solution of the first order differential equation

$$(d_t - \mathcal{N})\mathcal{U}(0,t) = 0$$

satisfying $\mathcal{U}(0,0) = \mathbb{1}$.

One can also check that under a gauge transformation the connection $\mathcal{N}$ transforms as follows:

$$\delta_\epsilon \mathcal{N} = -[d_t - \mathcal{N}, \epsilon^a T_a].$$

Hence the Wilson loop is gauge-invariant.

5.2 Wilson loops and the equivariant derived category

In this section we reformulate the data involved in the construction of a Wilson loop observable in purely holomorphic terms. We will see that Wilson loops can be naturally regarded as objects of a category which is a deformation of the equivariant derived category of coherent sheaves on X. This deformation apparently is required for the existence of a nontrivial braided monoidal structure on the category of Wilson loops. In the case of flat X, the deformed category should be equivalent to the derived category of representations of a quantum supergroup.

Let $(A, d) = (\Omega^{0,\bullet}, \bar{\partial})$ be the Dolbeault complex, regarded as a $\mathbb{Z}_2$-graded differential algebra. The algebra A has a Poisson bracket $\{,\}$ coming from the holomorphic symplectic form Ω. This bracket is even and d is a derivation with respect to it, i.e.

$$d\{f_1, f_2\} = \{df_1, f_2\} \pm \{f_1, df_2\}.$$

The triple $(A, d, \{,\})$ is a differential Poisson algebra.

The complexification $\mathfrak{g}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ acts on A by holomorphic vector fields $W_a = V_a^I \partial_I$, which we can regard as even derivations of A commuting with d. They satisfy

$$[W_a, W_b] = f_{ab}^c W_c.$$

These derivations are Hamiltonian, in the sense that there exist even elements $\mu_{+a} \in A$ (the moment maps) such that

$$W_a(f) = -\{\mu_{+a}, f\}, \quad \forall f \in A.$$

The moment maps are required to satisfy

$$\{\mu_{+b}, \mu_{+c}\} = -f_{bc}^a \mu_{+a}.$$

It is also required that μ_{+a} is isotropic with respect to an invariant metric κ^{ab} on the dual vector space of $\mathfrak{g}_{\mathbb{C}}$:

$$\kappa^{ab} \mu_{+a} \mu_{+b} = \mu_+ \cdot \mu_+ = 0.$$

The space of sections of the graded vector bundle $E \otimes \Omega^{0,\bullet}$ can be regarded as a graded module M over A. We denote by $f \bullet m$ the action of $f \in A$ on $m \in M$. The operator $D : M \to M$ is an odd derivation of M, i.e.

$$D(f \bullet m) = df \bullet m \pm f \bullet Dm, \quad \forall m \in M, \ \forall f \in A.$$

The operators $\mathcal{W}_a$ on the space of smooth sections of E make M into a $\mathfrak{g}_{\mathbb{C}}$-equivariant A-module. That is, M is an $\mathfrak{g}_{\mathbb{C}}$-module, and this module structure is compatible with the $\mathfrak{g}_{\mathbb{C}}$-module structure on A in the following sense:

$$\mathcal{W}_a(f \bullet m) = W_a(f) \bullet m + f \bullet \mathcal{W}_a(m), \quad \forall m \in M, \ \forall f \in A.$$

Finally, D satisfies the following two conditions:

$$[D, \mathcal{W}_a] = 0, \tag{31}$$

$$D^2 = \kappa^{ab} \mu_{+a} \mathcal{W}_b. \tag{32}$$

If we replace (32) with a condition $D^2 = 0$, then the triple $(M, D, \mathcal{W}_a)$ becomes an equivariant differential graded module over the supercommutative DG-algebra (A, d, W_a) with a $\mathfrak{g}_\mathbb{C}$-action. Note that the Poisson bracket, the moment maps, and an invariant metric on $\mathfrak{g}_\mathbb{C}^*$ are not needed to define such a module. But these data are needed to define the deformed category whose objects label the Wilson loops in the CSRW model.

Let us say a few words about morphisms in the deformed category. From the physical point of view these are observables inserted at the joining point of two Wilson lines labeled by objects $(M_1, D_1, \mathcal{W}_{1a})$ and $(M_2, D_2, \mathcal{W}_{2a})$. A morphism is therefore an A-module morphism $\phi : M_1 \to M_2$. This space has a natural $\mathbb{Z}_2$ grading. BRST operator Q defines an odd derivation D_{12} on the space of morphisms:

$$D_{12}\phi = D_2 \circ \phi \pm \phi \circ D_1.$$

This derivation is not nilpotent, rather

$$D_{12}^2\phi = \kappa^{ab}\mu_{+a}\left(\mathcal{W}_{2b} \circ \phi - \phi \circ \mathcal{W}_{1b}\right).$$

It is nilpotent on the subspace of equivariant morphisms, i.e. those ϕ for which the expression in parentheses on the r.h.s. of the above equation identically vanishes. Thus it is possible to define a differential-graded category whose objects are as above, morphisms are equivariant morphisms of A-modules, and the differential on the space of morphisms is D_{12}.

From the physical viewpoint this prescription for computing the space of observables is not quite correct since it does not include the Faddeev-Popov ghosts. The correct prescription is to tensor the space of morphisms with $\bigwedge^\bullet \mathfrak{g}_\mathbb{C}^*$ (the algebra of ghost fields) and compute the cohomology of the operator $\hat{Q}$. It has the form

$$\hat{Q} = D_{12} + Q_{CE} + \mu_{+a}\kappa^{ab}\frac{\partial}{\partial c^b},$$

where Q_{CE} is the Chevalley-Eilenberg differential corresponding to the $\mathfrak{g}_\mathbb{C}$-module $\mathrm{Hom}_A(M_1, M_2)$. One can check that $\hat{Q}^2 = 0$.

Replacing Q with $\hat{Q}$ should be thought of as passing to the derived version of the category. Indeed, if we formally set $\kappa^{ab} = 0$, $(M, D, \mathcal{W}_a)$ becomes the usual equivariant DG-module over (A, d, W_a), and tensoring $\mathrm{Hom}_A(M_1, M_2)$ with the ghost algebra and adding Q_{CE} to the differential D_{12} is the standard way to get a free resolution of the $\mathfrak{g}_\mathbb{C}$-module $\mathrm{Hom}_A(M_1, M_2)$.

Let us make a few comments about the deformed category. First of all, since D is an odd derivation of M, D^2 is an even endomorphism of the module M. Although $\mathcal{W}_a$ is not an endomorphisms of M, the combination $\kappa^{ab}\mu_{+a}\mathcal{W}_b$ is, thanks to the condition $\mu_+ \cdot \mu_+ = 0$. Indeed, if f is an arbitrary element of A, then

$$[\kappa^{ab}\mu_{+a}\mathcal{W}_b, f] = \kappa^{ab}\mu_{+a}[\mathcal{W}_b, f] = \kappa^{ab}\mu_{+a}W_b(f) = -\kappa^{ab}\mu_{+a}\{\mu_{+b}, f\} = -\frac{1}{2}\{\mu_+ \cdot \mu_+, f\} = 0.$$

Second, the two conditions (31,32) are compatible:

$$0 = [D^2, \mathcal{W}_c] = [\kappa^{ab}\mu_{+a}\mathcal{W}_b, \mathcal{W}_c] = \kappa^{ab}\left(\mu_{+a}[\mathcal{W}_b, \mathcal{W}_c] - [\mathcal{W}_c, \mu_a]\mathcal{W}_b\right) = \\ = \kappa^{ab}\left(f^d_{bc}\mu_{+a}\mathcal{W}_d + \{\mu_{+c}, \mu_{+a}\}\mathcal{W}_b\right) = \left(\kappa^{ab} f^d_{bc} + \kappa^{db} f^a_{bc}\right)\mu_{+a}\mathcal{W}_d = 0. \quad (33)$$

In the second line we used the $\mathfrak{g}$-invariance of κ^{ab}.

Third, the fact that the Poisson bracket came from a symplectic form was not important. Thus the deformation of the equivariant derived category makes sense in the context of Poisson manifolds. Furthermore, the condition $\mu_+ \cdot \mu_+ = 0$ can be relaxed to the condition that $\mu_+ \cdot \mu_+$ have vanishing Poisson brackets with any element of A. In the case when the Poisson bracket comes from a symplectic form this generalization is not very significant, since the Poisson center of A consists of constants. It becomes interesting when the Poisson bracket is degenerate.

Fourth, there is a $\mathbb{Z}$-graded version of the story. If A and M are $\mathbb{Z}$-graded and $d : A \to A$ and $D : M \to M$ have degree 1, then for the condition (32) to make sense μ_{+a} must have degree 2. Since W_a has degree 0, the Poisson bracket must have degree -2. Such a situation is realized, for example, when A is the Dolbeault complex of the cotangent bundle of a complex manifold Y, provided that we put the linear coordinates on the fibers in degree 2. Above we dealt with a special case of this, namely $Y = G_{\mathbb{C}}/B$.

Fifth, while the algebra A occurring in the CSRW model is the algebra of forms on a smooth complex manifold, the definition of the deformed equivariant derived category given above is purely algebraic and makes sense in greater generality.

Sixth, the deformation which takes us from the equivariant derived category of (A, d, W_a) to the category defined above should be thought of as a quantum deformation. Indeed, absorbing the Chern-Simons level k into the metric κ_{ab} we see that κ^{ab} is proportional to the Planck constant $1/k$ of the CSRW model. The classical limit is the limit $\kappa^{ab} \to 0$, in which case the category of Wilson loops reduces to the usual equivariant derived category.

5.3 Examples of Wilson loop obsevables

For flat X the CSRW model is equivalent to a super-Chern-Simons theory. Therefore there is a Wilson loop operator for every finite-dimensional representation R of the Lie superalgebra $\mathfrak{G}$:

$$\mathcal{W}_R = \mathrm{STr}_R\ P e^{\oint A^a M_a^{(R)} + \chi^I \lambda_I^{(R)}}.$$

Here $M_a^{(R)}$ and $\lambda_I^{(R)}$ are endomorphisms of a graded vector space R representing bosonic and fermionic generators of $\mathfrak{G}$. This is a special case of our general construction in Section 5.1. To see this, we take X to be the odd part of $\mathfrak{G}$. The even part $\mathfrak{g}$ of $\mathfrak{G}$ acts on X linearly. We take E to be a trivial vector bundle over X with fiber R. Since the origin of X is invariant under $\mathfrak{g}$, we can specify an action of $\mathfrak{g}$ on E by specifying its action on the fiber of E over the origin. We take the tautological action of $\mathfrak{g}$ on R (R is a representation of $\mathfrak{G}$ and therefore a representation of its even subalgebra $\mathfrak{g}$). Finally we let

$$D = \bar{\partial} - \phi^I \lambda_I^{(R)}.$$

It is not entirely clear if for flat X any object in the category of Wilson loops is isomorphic to an object of this special form. We conjecture this to be the case.

For a curved target space it is rather difficult to find nontrivial examples of BRST-invariant Wilson loops. One could start with equivariant holomorphic vector bundles or complexes of vector bundles and try to deform them. In the special case $X = T^*\mathbb{CP}^1$ we will now exhibit a family of Wilson loops for which deformation is unnecessary. Let us take E to be a holomorphic line bundle over X with a $SL(2,\mathbb{C})$-action and let D be the usual $\bar{\partial}$-connection on E (i.e. we let $\mathcal{T} = \mathcal{S} = 0$). If we want E to be an object of the category of Wilson loops, then the endomorphisms T_a must satisfy

$$\kappa^{ab}\mu_{+a}T_b = 0. \tag{34}$$

They must also satisfy

$$f_{ab}^c T_c + V_a^{\hat{J}} V_b^{\hat{K}} \mathcal{F}_{\hat{J}\hat{K}} = 0, \quad \partial_{\hat{P}} T_a = V_a^{\hat{K}} \mathcal{F}_{\hat{K}\hat{P}}. \tag{35}$$

Since $T^*\mathbb{CP}^1$ is simply-connected, we can specify E together with an $SU(2)$-invariant connection by specifying an $SU(2)$-invariant $(1,1)$ form $\mathcal{F}$ whose periods are integral multiples of 2π. We take

$$\mathcal{F} = (-i)n\hat{J}, \quad n \in \mathbb{Z}$$

where $\hat{J}$ is a Kähler form on $T^*\mathbb{CP}^1$ normalized so that the integral of $\hat{J}$ over the zero section of $T^*\mathbb{CP}^1$ is 2π. The corresponding line bundle $E = \mathcal{L}^n$ on $T^*\mathbb{CP}^1$ restricts to $\mathcal{O}(n)$ on $\mathbb{CP}^1$. Given this $\mathcal{F}$ the endomorphisms T_a are uniquely determined:

$$T_a = (-i)n\hat{\mu}_{3a},$$

where $\hat{\mu}_{3a}$ is the moment map for $\hat{J}$. The condition $\mu_+ \cdot T = 0$ is satisfied thanks to (17). Note that at the point $z = b = 0$ we have

$$T_1 = T_2 = 0, \quad T_3 = (-i)n.$$

That is, the point $z = b = 0$ is fixed by a $U(1)$ subgroup of $SU(2)$, and the fiber of the line bundle $\mathcal{L}^n$ over this point transforms in the representation with charge n. By $SU(2)$ symmetry this is true for any other point on the zero section of $T^*\mathbb{CP}^1$: the fiber of $\mathcal{L}^n$ over a point transforms in a charge n representation of the $U(1)$ subgroup preserving this point.

Let us call W_n the Wilson loop corresponding to the line bundle $\mathcal{L}^n$. At the classical level it is clear that the product of W_n and W_m is W_{n+m}. There may be no quantum corrections to this result, since the Wilson loop operator W_n cannot be deformed. This follows, for example, from the fact that the endomorphisms of W_n regarded as an object of the equivariant derived category of $T^*\mathbb{CP}^1$ are the same as the endomorphisms of the trivial line bundle. This implies that there are no endomorphisms of W_n with ghost number one whose descendants could be used to construct an infinitesimal deformation of W_n.

So far, the Chern-Simons level k did not play any role in the discussion. One place where it shows up is in the braiding properties of the Wilson loops. The braiding phase is computed by taking Wilson loops W_{n_1} and W_{n_2} along the closed curves γ_1 and γ_2 in R^3 with linking number one and computing the correlator $\langle \mathsf{W}_{n_1}\mathsf{W}_{n_2}\rangle$. Using the well-known property of the Green-function $G_{\mu\nu}$ [4] we find[9] at leading order in $1/k$ expansion the phase $e^{\frac{\pi i n_1 n_2}{k}}$ plus BRST-exact terms. The phase is essentially the same as in ordinary Chern-Simons perturbation theory for Wilson loops, with generators of the Lie algebra in a particular representation replaced with the endomorphisms T_a. We will see below that there can be no corrections to this phase at higher order in perturbation theory.

Another finite-k effect is a periodic identification among Wilson loops: W_n is isomorphic to W_{n+2k} in the category of Wilson loops. To see this, it is sufficient to exhibit an invertible morphism between W_{2k} and the trivial Wilson loop corresponding to a trivial line bundle on $T^*\mathbb{CP}^1$. The space of such morphisms can be thought of as the space of local observables which can be inserted at the endpoint of W_{2k} (which is now a Wilson line rather than a Wilson loop). Equivalently, by state-operator correspondence in the CSRW theory, it is the space of states of the theory on the space-time of the form $S^2 \times \mathbb{R}$, with a Wilson line W_{2k} inserted at $\{p\} \times \mathbb{R}$ where p is point on S^2.

The quantization is simplified by the fact that all fermionic fields as well as the bosonic field b parameterizing the fiber direction in $T^*\mathbb{CP}^1$ are massive, and the only matter field zero mode is that of the field z which parameterizes the zero section of $T^*\mathbb{CP}^1$. Further, with all massive fields in their ground state, the Gauss law constraint in the presence of a Wilson loop reads

$$\frac{k}{2\pi}\kappa_{ab}F^b = iT_a\delta^2(p) = n\hat{\mu}_{3\ a}(z)\delta^2(p),$$

where F is the curvature of the G-connection A. Thus the gauge field is also determined by z. We also see that for $n \neq 0$ there is a magnetic flux on S^2 proportional to n. More precisely,

[9] $\langle A^a_\mu(x_1)A^b_\nu(x_2)\rangle = -\hbar G_{\mu\nu}(x_1,x_2)\kappa^{ab}, \quad \oint_{\gamma_1} dx_1^\mu \oint_{\gamma_2} dx_2^\nu G_{\mu\nu}(x_1,x_2) = 1.$

for a fixed z the gauge group is broken down to a $U(1)$ (the stabilizer of z), and the Gauss law constraint says that the gauge bundle reduces to this $U(1)$ subgroup and its first Chern class is $n/2k$. Clearly, this makes sense only if n is an integer multiple of $2k$, $n = 2km$.

The conclusion is that we need to quantize $\mathbb{CP}^1$ parameterized by the zero mode of z. The symplectic form arises both from the Chern-Simons part of the action and the Wilson line. It is clear that it will be an integer multiple of the curvature 2-form of the line bundle $\mathcal{O}(1)$. If the coefficient is l, then quantization gives $l+1$-dimensional space of states which furnishes an irreducible representation of $SU(2)$. On the other hand, Gauss law constraint guarantees that this space will be an $SU(2)$-singlet. This means that the Chern-Simons contribution must exactly cancel the contribution from the Wilson loop, so that $l = 0$. (One can verify this explicitly). This proves that there is a unique monopole operator on the endpoint of any Wilson loop W_n such that n is divisible by $2k$. The invertibility of this morphisms then follows from general axioms of 3d TFT. Alternatively, it is clear that when an open Wilson line terminated by monopoles shrinks to a point, the monopoles just annihilate each other and one is left with the identity operator in the bulk theory.

Note that the phase arising from braiding must be invariant under $n \mapsto n + 2k$. This rules out any higher-order perturbative corrections to the phase computed above.

6 Concluding remarks

In the previous section we have determined the category of Wilson loop operators in the general CSRW model and provided some examples of Wilson loops. On general grounds, we expect this category to be a braided monoidal category. It would be very interesting to compute the braiding, which is bound to be nontrivial because of Chern-Simons terms. The most interesting case is that of $X = T^*(G_{\mathbb{C}}/B)$, since this theory is similar in many respects to the ordinary Chern-Simons theory. In particular, it has no nontrivial local operators and its partition function is finite and provides new invariants of three-manifolds. It would be very interesting to study the structure of the perturbation series in this and other CSRW models, since it provides new solutions of the IHX relation [13]. One might also speculate that the knot invariants arising from the Wilson loop correlators in the CSRW model are related to quantum group knot invariants at non-primitive roots of unity.

7 Appendix A

Here we explain how to derive the BRST transformations (2) of the CSRW model by twisting supersymmetry transformations in $N=4$ $D=3$ superconformal theory constructed by Gaiotto and Witten. For simplicity we do this for a flat target X. As we explained in section 2, BRST transformations (2) also work for a curved target with appropriate moment maps.

Let $\theta_A^{\dot{B}\ \alpha}$ be a parameter of supersymmetry transformation, where $A(\dot{B})$ runs over a doublet of $SU(2)_R(SU(2)_N)$, and α runs over a doublet of the Lorentz symmetry $SU(2)_L$. The supersymmetry transformations of the Gaiotto-Witten model are

$$\delta Q_A^{\bar{I}} = \theta_A^{\dot{B}\ \alpha}\lambda^{\bar{I}}_{\dot{B}\ \alpha} \tag{36}$$

$$\delta\lambda^{\bar{I}}_{\dot{A}\alpha} = \theta^{B\ \beta}_{\dot{A}}\sigma^{\mu}_{\alpha\beta}D_\mu Q^{\bar{I}}_B + \frac{1}{3}\theta^B_{\dot{A}\ \alpha}V^{\bar{I}\ C}_a\mu_{b\ CB}\kappa^{ab}$$

$$\delta(A^a_\mu)\sigma^\mu_{\alpha\beta} = \kappa^{ab}\theta^{A\dot{B}}_{(\alpha}\lambda^{\bar{I}}_{\beta)\dot{B}}\Omega_{\bar{I}\bar{J}}V^{\bar{J}}_{b\ A}.$$

Here

$$Q_1^{\bar{I}} = \phi^{\bar{I}}, \quad Q_2^{\bar{I}} = 2\Omega^{\bar{I}\bar{J}}g_{\bar{J}K}\phi^K, \quad V_1^{a\ \bar{I}} = V^{a\ \bar{I}}, \quad V_2^{a\ \bar{I}} = 2\Omega^{\bar{I}\bar{J}}g_{\bar{J}K}V^{a\ K},$$

$$\mu_{a\ 11} = 2\mu_{a\ -}, \quad \mu_{a\ 12} = -2i\mu_{a\ 3}, \quad \mu_{a\ 22} = 2\mu_{a\ +}.$$

We work in conventions $\Omega = \frac{1}{2}\Omega_{IJ}d\phi^I\wedge d\phi^J$, $\quad J = ig_{I\bar{J}}d\phi^I\wedge d\phi^{\bar{J}}$ so that hyper-Kähler structure implies $\Omega^{\bar{I}\bar{J}} = -\frac{1}{4}g^{\bar{I}K}\Omega_{KL}g^{L\bar{J}}$ and for flat X we have $g_{I\bar{J}} = \frac{1}{2}\delta_{I\bar{J}}$.

We twist by identifying $SU(2)_L$ and $SU(2)_N$. Then

$$\theta_1^{\dot{A}\ \beta} = \theta^{2\ \dot{A}\ \beta} = -\theta_{BRST}\epsilon^{\dot{A}\ \beta}.$$

We may set all other supersymmetry variation parameters to zero, keeping only the BRST parameter θ_{BRST}. Then we write fermions in terms of the fields of the twisted model as

$$\lambda^{\bar{I}}_{\dot{A}\ \alpha} = \epsilon_{\dot{A}\ \alpha}\eta^{\bar{I}} + 2\sigma^\mu_{\dot{A}\ \alpha}\Omega^{\bar{I}\bar{J}}g_{\bar{J}K}\chi^K_\mu \tag{37}$$

and plug (37) into (36) to obtain BRST transformation (2).

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